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Weak Solutions of the Boltzmann Equation Without Angle Cutoff

Carlo Cercignani[1]

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The definition of the concept of weak solution of the nonlinear Boltzmann equation, recently introduced by the author, is used to prove that, without any cutoff in the collision kernel, the Boltzmann equation for Maxwell molecules in the one-dimensional case has a global weak solution in this sense. Global conservation of energy follows.

KEY WORDS: Boltzmann equation; Energy conservation; Global solution.

1. INTRODUCTION

In 1989, DiPerna and Lions[(8)] used various previous results and remarks, together with their new concept of renormalized solution to prove the first general global existence theorem for the Boltzmann equation in the nonhomogeneous case. It was soon clear[(2)] that solutions depending on just one space variables are special in the sense that one may hope to obtain existence in a more traditional sense; the final step was recently performed by the author[(4)], who eliminated a truncation for small relative speeds in the collision term.

Here we are concerned with the initial value problem for the nonlinear Boltzmann equation for Maxwell molecules without cutoff, when the solution depends on just one space coordinate which might range from $-\infty$ to $+\infty$ or from 0 to 1 (with periodicity boundary conditions); for definiteness we stick to the latter case. Easy modifications, in the vein of ref. 6, are necessary to deal with the case of different boundary conditions. The x-, y- and z- component of the velocity $\mathbf{v} \in \mathbf{R}^3$

[1] Dipartimento di Matematica, Politecnico di Milano, Piazza L. da Vinci 32, 20133 Milano, Italy.

will be denoted by ξ, η and ζ respectively, and the equation reads

$$\frac{\partial f}{\partial t} + \xi \frac{\partial f}{\partial x} = Q(f, f) \tag{1.1}$$

with

$$Q(f, f)(x, \mathbf{v}, t) = \int \int B(\mathbf{n} \cdot (\mathbf{v} - \mathbf{v}_*), |\mathbf{v} - \mathbf{v}_*|)\,(f' f'_* - f f_*) \sin\theta \, d\theta \, d\phi \, d\mathbf{v}_*. \tag{1.2}$$

$$\mathbf{v}' = \mathbf{v} - \mathbf{n}[\mathbf{n} \cdot (\mathbf{v} - \mathbf{v}_*)]$$

$$\mathbf{v}'_* = \mathbf{v}_* + \mathbf{n}[\mathbf{n} \cdot (\mathbf{v} - \mathbf{v}_*)]$$

For a detailed explanation of the structure of the collision term, see ref. 3, 7, or 8. The angles θ and ϕ are the polar and azimuthal angles of the collision parameter $\mathbf{n} \in S^2$ relative to a polar axis in direction $\mathbf{V} = \mathbf{v} - \mathbf{v}_*$.

We introduce as in[(4)] the weak form of the collision term, $Q(f, f)$. We shall henceforth use the latter notation for the operator defined by:

$$\int_{[0,T]\times[0,1]\times\mathbf{R}^3} Q(f, f)(x, \mathbf{v}, t)\varphi(x, \mathbf{v}, t) d\mathbf{v}\, dx\, dt$$

$$= \frac{1}{2} \int_{[0,T]\times[0,1]\times\mathbf{R}^3\times\mathbf{R}^3\times S^2} B(\mathbf{n} \cdot (\mathbf{v} - \mathbf{v}_*), |\mathbf{v} - \mathbf{v}_*|)\,(\varphi' + \varphi'_* - \varphi - \varphi_*) f f_* \, d\mu \, dt\,. \tag{1.3}$$

for any test function $\varphi(x, \mathbf{v}, t)$ which is twice differentiable as a function of $\mathbf{v}$ with second derivatives uniformly bounded with respect to x and t. In Eq. (1.4) we have used the notation

$$d\mu = \sin\theta \;\, d\theta \, d\phi \, d\mathbf{v}_* d\mathbf{v} \, dx \tag{1.4}$$

We remark that for classical solutions the above definition is known to be equivalent to that in (1.2). The main reason for introducing it is that it may produce weak solutions (as opposed to renormalized solutions in the sense of DiPerna and Lions[(8)]) even if the collision term is not necessarily in L^1. This also avoids cutting off the small relative speeds, as done in[(6)].

For a function f to be a weak solution of the Boltzmann equation, it must satisfy Eq. (1.1), where the derivatives in the left hand side are distributional derivatives and the right hand side has been defined above.

We assign an initial value $f(x, \mathbf{v}, 0) = f_0(x, \mathbf{v})$, and we shall assume that $f_0 \in L^1_+([0, 1] \times \mathbf{R}^3)$ with the normalization

$$\int \int f_0 \, dx \, d\mathbf{v} = 1. \tag{1.5}$$

The association of the solution with the weak formulation is standard. The objective of this paper is to show that the initial value problem for the Boltzmann equation without angular cutoff has a global weak solution in the sense defined above. The main step in proving this is a proof that collision term $Q(f, f)$ is such that the expression in Eq. (1.3) is finite.

2. THE COLLISION TERM FOR WEAK SOLUTIONS IN THE CASE OF NONCUTOFF POTENTIALS

In this section we want to prove that the definition of Section 1 makes sense for inverse power potentials without introducing Grad's angular cutoff, as hinted at in the previous paper[4]. To this end we consider the following identity:

$$\begin{aligned}\int_0^1 ds \int_0^1 dt \frac{\partial^2}{\partial s \partial t}[\varphi(\mathbf{v} + s(\mathbf{v}' - \mathbf{v}) + t(\mathbf{v}_* - \mathbf{v}'))] \\ = \int_0^1 ds \left\{ \frac{\partial}{\partial s}[\varphi(\mathbf{v} + s(\mathbf{v}' - \mathbf{v}) + (\mathbf{v}_* - \mathbf{v}'))] - \frac{\partial}{\partial s}[\varphi(\mathbf{v} + s(\mathbf{v}' - \mathbf{v}))] \right\} \\ = \varphi(\mathbf{v}_*) - \varphi(\mathbf{v}') - \varphi(\mathbf{v}'_*) + \varphi(\mathbf{v}) \end{aligned} \tag{2.1}$$

Hence

$$\varphi(\mathbf{v}) + \varphi(\mathbf{v}_*) - \varphi(\mathbf{v}') - \varphi(\mathbf{v}'_*) = \int_0^1 ds \int_0^1 dt \sum_{i,j=1}^{3} \frac{\partial^2 \varphi}{\partial v_i \partial v_j}(v'_i - v_i)(v^*_j - v'_j) \tag{2.2}$$

If K is an upper bound for the second derivatives, we obtain the following estimate

$$|\varphi(\mathbf{v}) + \varphi(\mathbf{v}_*) - \varphi(\mathbf{v}') - \varphi(\mathbf{v}'_*)| \le 9K|\mathbf{v}' - \mathbf{v}||\mathbf{v}^* - \mathbf{v}'| \le 9K|\mathbf{V}||\mathbf{n} \cdot \mathbf{V}| \tag{2.3}$$

Hence if the kernel B diverges for $\theta = \pi/2$, but $B \cos\theta$ is integrable, then the integral with respect to θ does not diverge. We recall that, if the intermolecular force varies as the n-th inverse power of the distance, then

$$B(\mathbf{n} \cdot (\mathbf{v} - \mathbf{v}_*), |\mathbf{v} - \mathbf{v}_*|) = B(\theta)|\mathbf{V}|^{\frac{n-5}{n-1}} \tag{2.4}$$

where $B(\theta)$ is a non-elementary function of θ which for θ close to $\pi/2$ behaves as the power $-(n+1)/(n-1)$ of $|\pi/2 - \theta|$. In particular, for $n = 5$ one has the Maxwell molecules, for which the dependence on V disappears.

We conclude that for power-law potentials, $B \cos\theta$ behaves as the power $-2/(n-1)$ of $|\pi/2 - \theta|$ and the definition of a weak solution given in Section 1 makes sense for $n > 3$.

Henceforth we shall consider just Maxwell molecules, for which we state the main result of this section as

Lemma 2.1 *The following estimate holds*

$$\left|\int_{[0,T]\times[0,1]\times\mathbf{R}^3} Q(f,f)(x,\mathbf{v},t)\varphi(x,\mathbf{v},t)d\mathbf{v}\,dx\,dt\right|$$

$$\leq \beta_0 K \int_{[0,T]\times[0,1]\times\mathbf{R}^3\times\mathbf{R}^3} |\mathbf{V}|^2 f f_* d\mathbf{v}\,d\mathbf{v}_*\,dx\,dt\,. \tag{2.5}$$

where K is un upper bound for the second derivatives of ϕ and β_0 a constant that only depends on molecular parameters.

3. BASIC ESTIMATES

We recall from a previous paper[4] that if we cutoff the values of θ close to $\pi/2$ then there is a weak solution in the sense defined in Section 1.

Theorem 3.1. *Let $f_0 \in L^1(\mathbf{R}\times\mathbf{R}^3)$ be such that*

$$\int f_0(\cdot)(1+|\mathbf{v}|^2)d\mathbf{v}\,dx < \infty; \qquad \int f_0|\ln f_0(.)|d\mathbf{v}\,dx < \infty. \tag{3.1}$$

Also, assume that the collision kernel for Maxwell molecules B is cutoff for $|\theta-\pi/2|\leq\epsilon$ ($\epsilon>0$). Then there is a weak solution $f(x,\mathbf{v},t)$ of the initial value problem (1.1), (1.4)*, such that $f\in C(\mathbf{R}_+, L^1(\mathbf{R}\times\mathbf{R}^3))$, $f(.,0)=f_0$. This solution conserves energy globally.*

We now set out to prove the crucial estimates for the solution of the initial value problem and for the collision term. It is safe to assume that we deal with a sufficiently regular solution of the problem, because this can always be enforced by truncating the collision kernel and modifying the collision terms in the way described in earlier work, in particular in[8]. If we obtain strong enough bounds on the solutions of such truncated problems, we can then extract a subsequence converging to a renormalized solution in the sense of DiPerna and Lions; and the bounds which we do get actually guarantee that this solution is then a solution in the weak sense defined above.

Consider now the functional

$$I[f](t) = \int_{x<y}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}(\xi-\xi_*)f(x,\mathbf{v},t)f(y,\mathbf{v}_*,t)\,d\mathbf{v}_*\,d\mathbf{v}\,dx\,dy \tag{3.2}$$

where the integral with respect to x and y is over the triangle $0\leq x<y\leq 1$. This functional was in the one-dimensional discrete velocity context first introduced by

Bony 1. The use of this functional is the main reason why we have to restrict our work to one dimension; no functional with similar pleasant properties is known, at this time, in more than one dimension (for a discussion of this point see a recent paper of the author[(5)]). Notice that if we have bounds for the integral with respect to x of $\rho = \int_{\mathbf{R}^3} f(x, \mathbf{v}, t)\, d\mathbf{v}$ and for

$$E(t) = \int_0^1 \int |\mathbf{v}|^2 f\, d\mathbf{v}\, dx,$$

then we have control over the functional $I[f](t)$.

A short calculation with proper use of the collision invariants of the Boltzmann collision operator shows that

$$\frac{d}{dt} I[f] = -\int_{[0,1]} \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} (\xi - \xi_*)^2 f(x, \mathbf{v}_*, t) f(x, \mathbf{v}, t)\, d\mathbf{v}\, d\mathbf{v}_*\, dx \qquad (3.3)$$

Notice that the first term on the right, apart from the factor $(\xi - \xi_*)^2$, has structural similarity to the collision term of the Boltzmann equation, and the integrand is nonnegative. This is the reason why the functional $I[f]$ is a powerful tool.

After integration from 0 to $T > 0$ and reorganizing,

$$\begin{aligned} &\int_0^T \int_{[0,1]} \int_{\mathbf{v}} \int_{\mathbf{v}_*} (\xi - \xi_*)^2 f(x, \mathbf{v}_*, t) f(x, \mathbf{v}, t)\, d\mathbf{v}\, d\mathbf{v}_*\, dx\, dt \\ &\quad = I[f](0) - I[f](T). \end{aligned} \qquad (3.4)$$

According to a previous remark, the right-hand side of (3.4) is bounded. Since the total energy is conserved, we have proved.

Lemma 3.2. *If f is a sufficiently smooth solution of the initial value problem given by* (1.1) *and* (1.4) *with initial value f_0, then*

$$\int_0^t \int_0^1 \int_{\mathbf{v}} \int_{\mathbf{v}_*} (\xi - \xi_*)^2 f(x, \mathbf{v}_*, \tau) f(x, \mathbf{v}, \tau)\, d\mathbf{v}\, d\mathbf{v}_*\, dx\, d\tau$$

are bounded.

The idea of the basic estimates was given in ref. 2; we will repeat some details here to make this paper self-contained.

We have now the following.

Lemma 3.3. *Under the above assumptions, we have, for the weak solutions of the Boltzmann equation for noncutoff Maxwell molecules:*

$$\int_{\mathbf{R}^3\times\mathbf{R}^3\times[0,T]\times[0,1]} |\mathbf{v}-\mathbf{v}_*|^2 f(x,\mathbf{v},t) f(x,\mathbf{v}_*,t)dt\,d\mu < K_0 \tag{3.7}$$

where K_0 is a constant, which only depends on the initial data (and molecular constants).

In fact, we can take $\varphi = \xi^2$ as a test function and remark that the contribution of the left hand side is bounded in terms of the initial data because $\xi^2 \le |\mathbf{v}|^2$. Hence the integral in the right hand side is also bounded. When computing this integral, we use as polar angles θ (the angle between $\mathbf{n}$ and $\mathbf{V}$) and ϕ (a suitable angle in the plane orthogonal to $\mathbf{V}$) so that the components n_i $(i = 1, 2, 3)$ of $\mathbf{n}$ are given by

$$n_1 = \frac{V_1}{V}\cos\theta - \frac{V_0}{V}\sin\theta\,\cos\phi$$

$$n_2 = \frac{V_2}{V}\cos\theta + \frac{V_1V_2}{VV_0}\sin\theta\,\cos\phi - \frac{V_3}{V}\sin\theta\,\sin\phi$$

$$n_3 = \frac{V_3}{V}\cos\theta + \frac{V_1V_3}{VV_0}\sin\theta\,\cos\phi + \frac{V_2}{V}\sin\theta\,\sin\phi$$

where V_i $(i = 1, 2, 3)$ are the components of $\mathbf{V}$ and $V_0 = \sqrt{V_2^2 + V_3^2}$. Then we have

$$\xi' = \xi - V_1\cos^2\theta + \frac{1}{2}V_0\sin 2\theta\,\cos\phi$$

$$\xi'_* = \xi_* + V_1\cos^2\theta - \frac{1}{2}V_0\sin 2\theta\,\cos\phi$$

We have:

$$\varphi(\mathbf{v}) + \varphi(\mathbf{v}_*) - \varphi(\mathbf{v}') - \varphi(\mathbf{v}'_*) = 2V_1^2\cos^2\theta$$

$$-V_1V_0\sin 2\theta\cos\phi - 2V_1^2\cos^4\theta - \frac{1}{2}V_0^2\sin^2 2\theta\cos^2\phi + 2V_0V_1\cos^2\theta\,\sin 2\theta\,\cos\phi$$

Then after integrating with respect to ϕ:

$$\int_{[0,T]\times[0,1]\times\mathbf{R}^3} Q(f,f)(x,\mathbf{v},t)\xi^2 d\mathbf{v}\,dx\,dt$$

$$= \int_{[0,T]\times[0,1]\times\mathbf{R}^3\times\mathbf{R}^3\times S^2} B(\theta)\Big\{\pi\big[-2V_1^2\cos^2\theta + 2V_1^2\cos^4\theta\big]$$

$$+\frac{\pi}{4}V_0^2\sin^2 2\theta\Big)ff_*d\mu\,dt$$

$$=\int_{[0,T]\times[0,1]\times\mathbf{R}^3\times\mathbf{R}^3\times S^2} B(\theta)\Big\{\frac{\pi}{4}\big(V^2-3V_1^2\big\}\Big)\sin^2 2\theta ff_*d\mu\,dt\,. \tag{3.9}$$

We can separate the contributions from the two terms, since they separately converge and obtain

$$\int_{[0,T]\times[0,1]\times\mathbf{R}^3} Q(f,f)(x,\mathbf{v},t)\xi^2 d\mathbf{v}\,dx\,dt$$
$$=-3B_0\int_{[0,T]\times[0,1]\times\mathbf{R}^3\times\mathbf{R}^3}(\xi-\xi_*)^2 ff_* d\mathbf{v}\,d\mathbf{v}_*\,dx\,dt$$
$$+B_0\int_{[0,T]\times[0,1]\times\mathbf{R}^3\times\mathbf{R}^3}|\mathbf{V}|^2 ff_* d\mathbf{v}\,d\mathbf{v}_*\,dx\,dt. \tag{3.10}$$

where if the force between two molecules at distance r is κr^{-5}, then

$$B_0=a\sqrt{\frac{\kappa}{2m^3}}\quad (a=1.3703\ldots). \tag{3.11}$$

The constant a was first computed by Maxwell[10]; the value given here was computed by Ikenberry and Truesdell[9]. Since we know that the left hand side of Eq. (3.10) is bounded and the first term in the right hand side is bounded, it follows that the last term is also bounded by a constant depending on initial data (and molecular constants, such as m and κ).

4. EXISTENCE OF WEAK SOLUTIONS FOR NONCUTOFF POTENTIALS

In order to prove the existence of a weak solution, we shall assume that this has been proved for Maxwell molecules with an angular cutoff[4], as stated in Theorem 3.1; actually to make the paper self-contained and the proof more explicit, we shall assume that the proof is available when a cutoff for small relative speed is introduced. In this case, in fact the proof immediately follows from the DiPerna-Lions existence theorem with the estimate of Lemma 3.4; it is enough to remark that a solution exists when we renormalize by division by $1+\epsilon f$ (f independent of $\epsilon>0$) and we case to the limit $\epsilon\to 0$ thanks to (3.7), which, of course, holds in the cutoff case as well.

In the noncutoff case we approximate the solution by cutting off the angles close to $\pi/2$ and the small relative speeds. In this way we can obtain a sequence f_n formally approximating the solution f whose existence we want to prove.

Lemma 4.1. *Let $\{f^n\}$ be a sequence of solutions to an approximating problem. There is a subsequence such that for each $T > 0$*

i) $\int f^n \, d\mathbf{v} \to \int f \, d\mathbf{v}$ *a.e. and in* $L^1((0, T) \times \mathbf{R}^3)$,

ii)

$$\int_{\mathbf{R}^3} |\mathbf{V}|^2 f_{n*} d\mathbf{v}_* \to \int_{\mathbf{R}^3} |\mathbf{V}|^2 f_* d\mathbf{v}_*$$

in $L^1((0, T) \times \mathbf{R}^3 \times B_R)$ *for all* $R > 0$, *and a.e.,*

iii)

$$g_n(x, t) = \frac{\int_{\mathbf{R}^3 \times \mathbf{R}^3} |\mathbf{V}|^2 f_n f_{n*} d\mathbf{v} \, d\mathbf{v}_*}{1 + \int f_n \, d\mathbf{v}} \to \frac{\int_{\mathbf{R}^3 \times \mathbf{R}^3} |\mathbf{V}|^2 f f_* d\mathbf{v} \, d\mathbf{v}_*}{1 + \int f \, d\mathbf{v}} = g(x, t) \tag{4.1}$$

weakly in $L^1((0, T) \times (0, 1))$.

Proof: (i) is immediate. (ii) uses an argument well-known in DiPerna-Lions proof with the estimate $\sup_n \int f_n (1 + |\mathbf{v}|^2) \, d\mathbf{v} < \infty$ to reduce the problem to bounded domains with respect to $\mathbf{v}_*$.

For (iii) we use (i) and the fact that f_n converges weakly, but the factor multiplying it in the integral converges a.e. because of (ii). □

Now we remark that $g_n(x, t)$ converges weakly to $g(x, t)$ and $\rho_n(x, t)$ converges a.e. to $\rho(x, t)$ and the integral $\int \rho_n g_n dx dt$ is uniformly bounded to conclude with the following Lemma:

Lemma 4.2. *Let $\{f_n\}$ be a sequence of solutions to an approximating problem. There is a subsequence such that for each $T > 0$*

$$\int_{(0,T)\times(0,1)\times\mathbf{R}^3\times\mathbf{R}^3} |\mathbf{V}|^2 f_n f_{n*} d\mu dt \to \int_{(0,T)\times(0,1)\times\mathbf{R}^3\times\mathbf{R}^3} |\mathbf{V}|^2 f f_* d\mu \, dt \tag{4.2}$$

We can now prove the following, basic result:

Lemma 4.3. *Let $\{f_n\}$ be a sequence of solutions to an approximating problem, weakly converging to f. There is a subsequence such that for each $T > 0$*

$$\int_{(0,T)\times(0,1)\times\mathbf{R}^3} \phi Q_n(f_n, f_n) dt dx d\mathbf{v} \to \int_{(0,T)\times(0,1)\times\mathbf{R}^3} \phi Q(f, f) dt \, dx \, d\mathbf{v} \tag{4.3}$$

where Q_n and Q are given by the weak form of the collision operator, as defined in Eq. (1.4).

Proof: In fact the integrand in the left hand side of Eq. (4.3) is dominated by the integrand of Eq. (4.2) which weakly converges and we can take the limit.

Thanks to this result, we can now pass to the limit in the approximating problem to obtain □

Theorem 4.4. *Let $f_0 \in L^1(\mathbf{R} \times \mathbf{R}^3)$ be such that*

$$\int f_0(\cdot)(1 + |\mathbf{v}|^2) d\mathbf{v}\, dx < \infty; \qquad \int f_0 |\ln f_0(.)| d\mathbf{v} dx < \infty. \tag{4.4}$$

Then there is a weak solution $f(x, \mathbf{v}, t)$ of the initial value problem (1.1), (1.4), such that $f \in C(\mathbf{R}_+, L^1(\mathbf{R} \times \mathbf{R}^3))$, $f(., 0) = f_0$. This solution conserves energy globally.

5. CONCLUDING REMARKS

We have proved existence of a weak solution of the nonlinear Boltzmann equation for Maxwell molecules, without any truncation on the collision kernel, in the one-dimensional case. To the best of our knowledge, this is the first result for the noncutoff Boltzmann equation. The solution conserves energy globally.

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REFERENCES

1. M. Bony, Existence globale et diffusion en théorie cinétique discrète. In R. Gatignol and Soubbarameyer, editors, *Advances in Kinetic Theory and Continuum Mechanics,* pp. 81–90 (Springer-Verlag, Berlin, 1991).
2. C. Cercignani, Weak solutions of the Boltzmann equation and energy conservation. *Appl. Math. Lett.* **8**: 53 (1995).
3. C. Cercignani, *Theory and Application of the Boltzmann equation* (Springer Verlag, New York, 1988).
4. C. Cercignani, Global weak solutions of the Boltzmann Equation. *J. Stat. Phys.* **118**: 333 (2005).
5. C. Cercignani, Estimating the solutions of the Boltzmann equation. Submitted to *J. Stat. Phys.* (2005).
6. C. Cercignani and R. Illner, Global weak solutions of the Boltzmann equation in a slab with diffusive boundary conditions. *Arch. Rational Mech. Anal.* **134**: 1 (1996).
7. C. Cercignani, R. Illner, and M. Pulvirenti, *The Mathematical Theory of Dilute Gases* (Springer Verlag, New York, 1994).

8. R. DiPerna and P.L. Lions, On the Cauchy problem for Boltzmann equations: Global existence and weak stability. *Ann. Math.* **130**: 321 (1989).
9. E. Ikenberry and C. Truesdell, On the pressures and the flux of energy in a gas according to Maxwell's kinetic theory, I. *J. Rat. Mech. Anal.* **5**: 1–54 (1956).
10. J. C. Maxwell, On the dynamical theory of gases. *Phil. Trans. Roy Soc. (London)* **157**: 49–88 (1866).